The Liar equilibrium in naked sovereign CDS trading : a financial economic approach

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MASTER 2 RECHERCHE MONNAIE, BANQUE, FINANCE

The Liar Equilibrium in Naked Sovereign CDS Trading: A Financial Economic Approach

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Master's thesis conducted under the direction of

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University year 2010-2011

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Abstract: This Master's thesis aims to study the impact of naked sovereign CDS trading by establishing a sequential trading model where the evolution of the market microstructure bid-ask spread is taken into account. The underlying-asset-covered positions as well as the naked positions are distinguished according to their different motifs in trading CDS. The main findings of our model suggests that the naked traders always prefer to deviate from the covered traders' equilibrium strategies in pursuit of the maximum profit, by manipulating following traders' belief and deceiving them into the wrong informational cascade. Consequently, an earlier start of a more possible wrong cascading effect occurs, resulting in a less accurate market estimation of the underlying country's default risk.

Key words: Sovereign CDS, covered and naked trading, microstructure, cascading effect, liar equilibrium.

« L’université de Paris 1 Panthéon Sorbonne n’entend donner aucune approbation, ni désapprobation aux opinions émises dans ce mémoire ; elles doivent être considérées comme propre à leur auteur. »

1. Introduction

The Credit Default Swaps (CDS) are designed to protect efficiently the credit lenders from the default risk of the underlying borrower. The CDS is a protection contract bought by credit lenders at the expense of a premium paid to the CDS seller. In case of a credit event[1], the buyer can recover his loss by a settlement delivered from the CDS seller, which makes the CDS similar to a put option.

These two financial instruments came out of hedging needs, but both have been developing for speculation desires. Although there is no evidence showing the existence of bubbles on the CDS markets, it is widely accepted that the market is now dominated by speculators rather than by hedgers. Not only because the global CDS market realized an exponential growth since the early 2000s to the eve of the fall of Lehman Brothers, but Lehman's own default also triggered a delivery of about 400 billion dollars of which 98.2% has been offset by opposite positions, proving the existence of simultaneous long and short position holders. Therefore, trading without having lent to the underlying borrower is called the 'naked' that is seriously blamed and is even suggested to be banned.

The most obvious argument against the naked CDS is the moral hazard arising when it is impossible to insure without an 'insure interest' - as in taking out life assurance on someone else's life.[2] Thus naked CDS trading is considered as a major accomplice for the European sovereign debt crisis starting since the early 2010; especially the Greek CDS spread that has been quadrupled during the last year. Nevertheless, we do not agree with this metaphor, because the beneficiary of the life

[1] The credit event that triggers a CDS is strictly defined according to the standard documents promulgated by the International Swaps and Derivatives Association (ISDA).
[2] Richard Portes, Ban naked CDS,

assurance must be the assured[3], so a different policy-holder cannot benefit from the death of the assured. On contrary, to benefit from the insured's default is what naked CDS traders are looking forward to; and this is feasible because naked CDS traders cannot singlehandedly force the underlying country to default.

Another criticism is that naked traders only bet according to their perception on the underlying country's default, and to pile on the agony, they might bet because they believe others might bet the same way. This will cause sharp changes of CDS premium that is regarded as a good estimator of a country's financial strength to repay his foreign loans, according to Flannery *et al.* (2010). Consequently, the estimation of the real default risk could be biased. So this master's thesis is interested in the impacts of the naked sovereign CDS trading and to what extent it influences the default of the underlying country.

The rest of this paper will be organized as follows. Section 2 summarizes relevant theoretical researches on the microstructure as well as on informational cascade. Section 3 describes our three-agent sequential trading model in details. Section 4 presents the main findings of our model and analyses the different equilibriums between covered and naked trading. Section 5 adds our point of view generated from the findings of the present model to the existing debate on naked CDS trading. Section 6 concludes.

2. Literature review

Although a number of criticisms have been made to naked CDS trading, yet there is no formal research studying it, neither empirical nor theoretical. Even if on a larger

[3] If not, the policy holder can murder the assured to benefit from the insurance gold, if the beneficiary is not the assured but himself.

horizon for all credit derivatives markets, most of the researches aim at either corporate CDS trading or empirical CDS pricing. So this thesis tries to enrich both the theoretical researches and the study on sovereign CDS, through a sequential trading model where a dynamic microstructure noise is integrated.

Acharya and John (2007) verify the insider trading on corporate CDS markets, as the number of relationship banks is increasing, who exclusively possess some bad news about the underlying firms. The existence of information asymmetry permits us using a microstructure model to study the market behavior. However, the theoretic microstructure model has not yet been used to analyze the CDS market, let alone the sovereigns. Thus, we can only refer to the researches on stock and options markets, as CDS resembles the put. Generally speaking, these microstructure models are classified by two criteria that whether private information concerns the future price or the volatility, and that whether trading is allowed on stock market or options market.

Based on Kyle (1985) who demonstrates that all private information will be incorporated into the latest price, Glosten and Milgrom (1985) develop the positive bid-ask price of which the width is related to the fraction of informed traders, and Easley and O'Hara (1987) investigate how trade size affects the price. Then option trading is introduced, Back (1993) shows the initially redundant options can cause the volatility of the underlying stock to become stochastic. Brennan and Cao (1996) find that both well and poorly informed traders gain extra welfare from trading more frequently. Easley *et al.* (1998) identify the informed trading on options market, since equity price changes induce hedge trading in options, and that particular options volumes can lead stock prices. John *et al.* (2000) argue that a high level of margin requirements enhances the informed traders' bias towards the spot market, while a low level weakens this bias; nevertheless the introduction of options increases the market informational efficiency.

Cherian and Jarrow (1998) then introduce the unmentioned volatility traders who only have private information on the volatility of the underlying assets. They prove that even if the underlying asset does not follow a lognormal distribution, the Black-Scholes formula is still satisfied on condition that the market believes it. Thus the volatility information is incorporated in the equilibrium price. Nandi (1999) concentrates only on the options market with volatility insiders. He finds that the level of implied volatility is increasing of the net options order flow to the market maker, resulting in a greater mispricing by Black-Scholes model.

Unlike the traditional way to distinguish between the informed and the uninformed, our model has only one homogenous group of traders who are uncertainly informed, instead of having the insider certainly informed of the uncertainty and the noise trader total trading at random. This equivalent change in microstructure point of view permits establishing a relationship with the informational cascade models whose impacts worry many economists in sovereign debt crisis.

Bikhchandani *et al.* (1992) introduced theoretically the notion of an informational cascade that an individual follows by having observed the behavior of the proceeding individuals, without regard to his own information. Though fragile, the informational cascade is easily to start with very little information because of the local conformity. Then this notion is largely used in financial economics. Ginblatt *et al.* (1995) and Wermers (1999) find that mutual funds tend to purchase past winning stocks, which is called momentum strategy; and that small stocks are largely preferred by growth-oriented funds. Alevy *et al.* (2007) find that although students are more Bayesian in decision making, professionals are not significantly outperformed, because they are better in recognizing the quality of information. Unlike these researches, the binary signals in this thesis are asymmetrical.

The interaction between both microstructure information asymmetry model and informational cascade model in the Master's thesis serves as a generalization of Allen and Gorton (1991) who point out the possibility for uninformed traders to singlehandedly realize an arbitrage by manipulating stock prices. Under the assumption that liquidity traders are prior to sell stocks for liquidity, they only used a numerical example to display the arbitrage.

3. Model descriptions

Our model separates covered and naked traders, and employs a comparative approach to distinguish the cascading effects on the sovereign CDS market without and with naked traders, so that the impacts of naked trading can be deduced. In this section, we are going to present the states of nature, the game rules and the payoffs of our model.

3.1 The states of nature

Unlike the options whose underlying asset is directly traded on stock market, the underlying of the sovereign CDS that is the asset value of a country cannot be traded. Investors are concerned of whether the asset value of the underlying country is sufficient to fully repay its debt at the expiration. By referring to Merton (1974) whose approach to price the CDS is widely used today, we assume:

Assumption 1 The evolution of country's asset value follows a Geometric Brownian Motion (GBM).

$$\frac{dV(t)}{V(t)} = \mu * dt + \sigma * dW(t) \tag{1}$$

$V(t)$ is the asset value of a country at time t, $W(t)$ is a Weiner process. μ represents a riskless rate of return, and σ expresses the volatility of the country's asset value, which we interpret as the volatility of the country's economic growth.

According to Ito's lemma, the cumulative return which in our model is the cumulated economic growth follows a logarithmic normal distribution:

$$\ln\frac{V(t)}{V(0)} \sim N\left(\left(\mu - \frac{1}{2}\sigma^2\right)t\,,\ \sigma^2 t\ \right) \tag{2}$$

The credit event occurs if the cumulated growth fails to meet a threshold[4] denoted by R. So the probability of default can be expressed as follows, where ε is a standard normal distribution.

$$\mathrm{Def} = \Pr\left(\ln\frac{V(t)}{V(0)} < R\right) = \Pr\left(\varepsilon < \frac{\ln\frac{R}{V(0)} - \left(\mu - \frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}} = -d_2\right) \tag{3}$$

Since that the risk-free rate and the time to expiration are considered as constant, the economic volatility σ is the only variable characterizing equation (2).

Proposition 1 The probability of default is positively related with the economic volatility.

$$\frac{\partial \mathrm{Def}}{\partial \sigma} > 0 \tag{4}$$

Proof: See in appendix.

In a mathematic point of view, a lower level of σ signifies that the cumulated growth follows a lognormal distribution with higher mean and thinner tail. So it is less likely to fail a certain threshold, hence the CDS should be less expensive. Under the positive correlation between the CDS premium and the volatility σ, we assume for theoretical conveniences that there are only two possible states of nature $\Lambda = \{\sigma_H, \sigma_L\}$, with σ_H standing for a country with high economic volatility while σ_L for low volatility; and the associated probability of default is respectively Def_H and Def_L.

[4] This threshold includes logically both the reimbursement to credit lenders and the paycheck for domestic workers.

The true state is denoted by $\lambda \in \Lambda$; and each state realizes with the same probability, that is $\Pr(\lambda = \sigma_H) = \Pr(\lambda = \sigma_L) = 0.5$.

3.2 Game rules

In our model the set of agent is defined as $I = \{i|i = 1,2, \dots, N\}$, and the first three as well as a risk neutral market maker[5] will be studied. For each agent the set of decision is identical and has only two choices $D_i = \{B, S\}$ with B meaning buying one unit of sovereign CDS while S meaning selling one unit. Although a CDS trader's action cannot be observed by others in an explicit way, a strongly reliable induction can be made from the historical prices of the CDS spread[6]. These two characteristics permit the existence of cascading effect in CDS trading.[7]

Three dates are considered in our model, respectively$\{t|t = 0,1, T\}$. σ is the volatility on the underlying nation's economic growth from t=1 to the expiration t=T[8], which could be either σ_H or σ_L at t=1. The present CDS price equals $C(\overline{\sigma})$ with $\overline{\sigma} = \frac{1}{2} * (\sigma_H + \sigma_L)$. Using a second-order Taylor expansion around $\overline{\sigma}$, we have the beneath relationship between the CDS prices[9]:

$$C(\sigma_H) = C(\overline{\sigma}) + \nabla(\sigma_H - \overline{\sigma}) = C(\sigma_L) + \nabla(\sigma_H - \sigma_L) \tag{5}$$

At t=0, the trader can receive a binary private signal and makes a decision by

5 Although the CDS market is an over-the-counter (OTC) market where buyers and sellers negotiate directly with each other, it cannot be forgotten the role of brokers who serve as a representative for both sides. So it is the broker who plays the role of market maker for he is responsible of proposing both purchase price and sell price to the real CDS investors.

6 Since the CDS spread is public information, it is observable for all market participants. The latest quotation listed must be the equilibrium price where the demand equals the supply. Therefore, if the first agent chooses to buy a CDS, the demand outweighs the supply, the premium increases as a result, and vice versa.

7 It is widely accepted that an informational cascade could occur only when the set of actions is limited and the actions should be observable.

8 The sovereign CDS contracts have different maturities, of which the most common is 5 years.

9 $\nabla = \frac{\partial C(\sigma)}{\partial \sigma}$ meaning the sensibility of the changes in CDS premium to those in the volatiliy of the underlying nation's economic growth.

maximizing his expected utility. Then the market maker should update the bid-ask spread whenever an action is fulfilled. This is repeated three times between t=0 and 1. Independently and identically distributed, the binary signals are denoted as $S_i = \{g, b\}$, of which a good signal g indicates that the underlying nation has a stable economic growth, whereas a bad signal b signifies that the nation is undergoing heterogeneous economic fluctuations. Table 1 summarizes the binary signals:

Table 1 Signal conditional probability

	$\Pr(S_i = g \mid \lambda)$	$\Pr(S_i = b \mid \lambda)$
$\lambda = \sigma_L$	Q	$1 - P$
$\lambda = \sigma_H$	$1 - Q$	P

Assumption 2 $P > Q > 0.5$, and $P - Q$ not too large[10].

If a country's economic growth is truly stable, it is more likely to observe a good signal: $\Pr(S_i = g \mid \lambda = \sigma_L) = Q > 0.5$, and vice versa for $\Pr(S_i = b \mid \lambda = \sigma_H) = P > 0.5$. However, the sovereign market is more sensitive to bad news, so bad signal occurs more often when an economic growth is stable than a good signal occurs when it is volatile, which implies $P > Q$. This signal asymmetry is different from all other informational cascade models where signals are symmetrical.

In addition, the uncertainty of the signal allows the market maker to fix the bid-ask spread so that the ex ante expected profit of one action is zero, since either true state of nature can realize at t=1 and each action could both gain and lose ex post. It is the same to the market maker in traditional microstructure models who offsets the gain of informed traders by the loss of noise traders. Moreover, a dynamic bid-ask price evolution is considered, which is respectively denoted by B_i^{θ} and A_i^{θ} the bid price and the ask price for agent i $\{i \mid i = 1,2,3\}$ known all the actions (θ) taken before.

[10] This signifies that the uncertainty degree of the two asymmetrical signals is alike.

3.3 The payoffs

Our model separates Covered traders who care about both his lending and the CDS at the expiration date, from naked traders who only try to buy CDS low and sell it high. Let $K = 1 - k$ with k the debt recovery rate[11] in case of a credit event, if the underlying country defaults, covered investor can get compensation from CDS seller at the expense of the premium: $\pi(B|Def) = K - A_i^{\theta}$; whereas $\pi(B|No\ Def) = -A_i^{\theta}$. On the other hand, CDS seller gains the premium without the underlying defaulting: $\pi(S|No\ Def) = B_i^{\theta}$; otherwise he compensates the protection buyer $\pi(S|Def) = B_i^{\theta} - K$. Let Ω be the whole information set of agent i out of which he can only make one choice, the profit of a covered trader can be expressed as follows:

$$\pi_i(D_i|\Omega) = \sum_{\lambda=\sigma_H,\sigma_L} \sum_{\Gamma=Def,No\ Def} \pi(D_i|\Gamma) * Pr(\Gamma|\lambda) * Pr(\lambda|\Omega) \qquad (6)$$

Assumption 3 The market is liquid enough; the CDS premium is free of arbitrage opportunities; and traders are all risk neutral[12].

Unlike covered traders who only make one decision between buying and selling CDS in our model, naked investors are obliged to have a pair of actions composed of two opposite positions whose profit is $B_j^{\theta} - A_i^{\theta}, i \neq j$. Otherwise he could suffer from heavy losses if for example the credit event happens and he only has short position. The pair of actions can be verified by the case of Lehman Brothers' CDS settlement after its bankruptcy that has been highlighted in section 1. The dynamic microstructure bid-ask prices consider the transaction cost twice[13] both at purchase and at liquidation, so that naked speculation becomes more constraining.

[11] The CDS premium is also expressed in percentage to the notional value of the contract; K equals the loss rate in the point of view of credit lenders.

[12] This implies that $u(\pi) = \pi$; so instead of comparing the utility, it is more direct to compare the payoffs.

[13] In traditional microstructure models based on information asymmetry, only the transaction cost at purchase is taken into account to calculate the profit; for the evolution of the microstructure bid-ask is not studied.

4. Model resolution

In this section we will determine respectively the equilibrium of only covered traders and that under the presence of naked traders. Through a comparison of the two equilibriums, we can find out the impacts of naked sovereign CDS trading.

4.1 The covered only equilibrium

In this subsection, we will firstly determine the decision making criteria for all covered traders. Then, we will demonstrate their optimal strategies through the Bayes' Rule. Finally, we will calculate the dynamic bid-ask prices for every trader.

4.1.1 Equilibrium conditions

Since covered traders in our model only makes one decision between buying and selling, they maximize the ex ante expected utility of his decision. If $\mathrm{Eu}(\mathrm{B}|\Omega) > \mathrm{Eu}(\mathrm{S}|\Omega)$[14], covered investors will buy CDS, and vice versa.

Lemma 1 The comparison between the expected utility of each action is equivalent to that of the conditional probability that the true state of nature realizes.

$$\mathrm{Eu}(\mathrm{B}|\Omega) \lesseqgtr \mathrm{Eu}(\mathrm{S}|\Omega) \text{ if and only if } \Pr(\sigma_\mathrm{H}|\Omega) \lesseqgtr \frac{1}{2} \text{ or } \Pr(\sigma_\mathrm{L}|\Omega) \gtreqless \frac{1}{2} \qquad (7)$$

Proof: See in appendix.

This is a generalized lemma for all the agents as we do not give any specific assumption to the information set Ω. This lemma makes the decision making much more direct and evident, with which we can demonstrate the optimal choices for all the covered traders. Having received a certain signal, each trader should deduce from his

[14] To recall that Ω represents all the information at the disposal of a trader before making the decision.

information set the conditional probabilities of each true state of nature realizing, of which a general formula is presented in lemma 2 under the following assumption.

Assumption 4 The actions made by all previous investors are independent to the signal received by the next trader.

Lemma 2 Generalized conditional probabilities on the entire information set where n= (1, 2, 3…n) and $0 \leq j \leq n-1$.

$$\Pr\left(\sigma_H \middle| \underbrace{B, B, \dots, B}_{j}, \underbrace{S, S, \dots, S}_{n-1-j}, b\right) = \frac{P^{j+1} * (1-P)^{n-1-j}}{P^{j+1} * (1-P)^{n-1-j} + (1-Q)^{j+1} * Q^{n-j-1}} \quad (8)$$

$$\Pr\left(\sigma_L \middle| \underbrace{B, B, \dots, B}_{j}, \underbrace{S, S, \dots, S}_{n-1-j}, g\right) = \frac{(1-Q)^{j} * Q^{n-j}}{(1-Q)^{j} * Q^{n-j} + P^{j} * (1-P)^{n-j}} \quad (9)$$

Proof: See in appendix.

4.1.2 Optimal strategies of the covered traders

According to lemma 2, every agent in our sequential trading model will calculate in turns his own conditional probabilities and hence make the ex ante optimal choice as lemma 1 suggests. The whole equilibrium of covered traders is presented in figure 1.

Proposition 2 The equilibrium strategies for covered traders are as follows:

1) Trader 1 has a separating strategy of which good signal makes him buy and bad signal makes sell CDS.
2) If the private signal of trader 2:

   Coincides with the first one's equilibrium strategy, he will follow.

   Contradicts the first one's equilibrium strategy, he will always prefer to sell a CDS.
3) If the two consecutive actions observed are:

   Identical, trader 3 will follow no matter what his private signal is.

Adverse, trader 3 will behave according to his own private signal.

Proof: See in appendix.

The signal of trader 1 dominates his decision, because it is the only hint for the real type of the underlying country. Good news implies that the underlying is more likely to realize a stable economic growth and hence a better repaying capacity, so the covered trader sells CDS believing ex ante that the credit event is less likely to happen; it is the same for bad news occurring. This result is consistent with that of other informational cascade models. However, we should notice that our asymmetrical signal $(P > Q)$ framework results in a higher reliance of the good signal and a lower reliance of the bad signal, compared to symmetrical signals $(P = Q)$[15]. That is $\Pr(\sigma_L|g)^{AS} > \Pr(\sigma_L|g)^{S}$ and $\Pr(\sigma_H|b)^{AS} < \Pr(\sigma_H|b)^{S}$[16]. This is because the bad news catches more attention of the sovereign CDS market, for lack of risk-controlling on sovereign debts.[17]

Having observed the action of trader 1, trader 2 follows (buying and selling) if his signal corresponds to the first one's equilibrium action. On contrary, if his signal contradicts the first one's equilibrium action, he will always prefer to sell a CDS. The first case is explained by the enhancing effect of two consecutive consistent signs, which in a mathematic sense is $\Pr(\sigma_L|S, g) > \Pr(\sigma_L|g)$,[18] as well as $\Pr(\sigma_H|B, b) > \Pr(\sigma_H|\, b)$. This finding is also consistent with that of other cascade theoretical models.

[15] 'AS' represents asymmetrical signals, whereas 'S' stands for symmetrical signals. In the rest of this dissertation where there is no such notation difference, all the probabilities are within the asymmetrical signal framework.

[16] Symmetrical framework demonstrated that $\Pr(\sigma_L|g)^S = Q$ and $\Pr(\sigma_H|b)^S$; therefore,

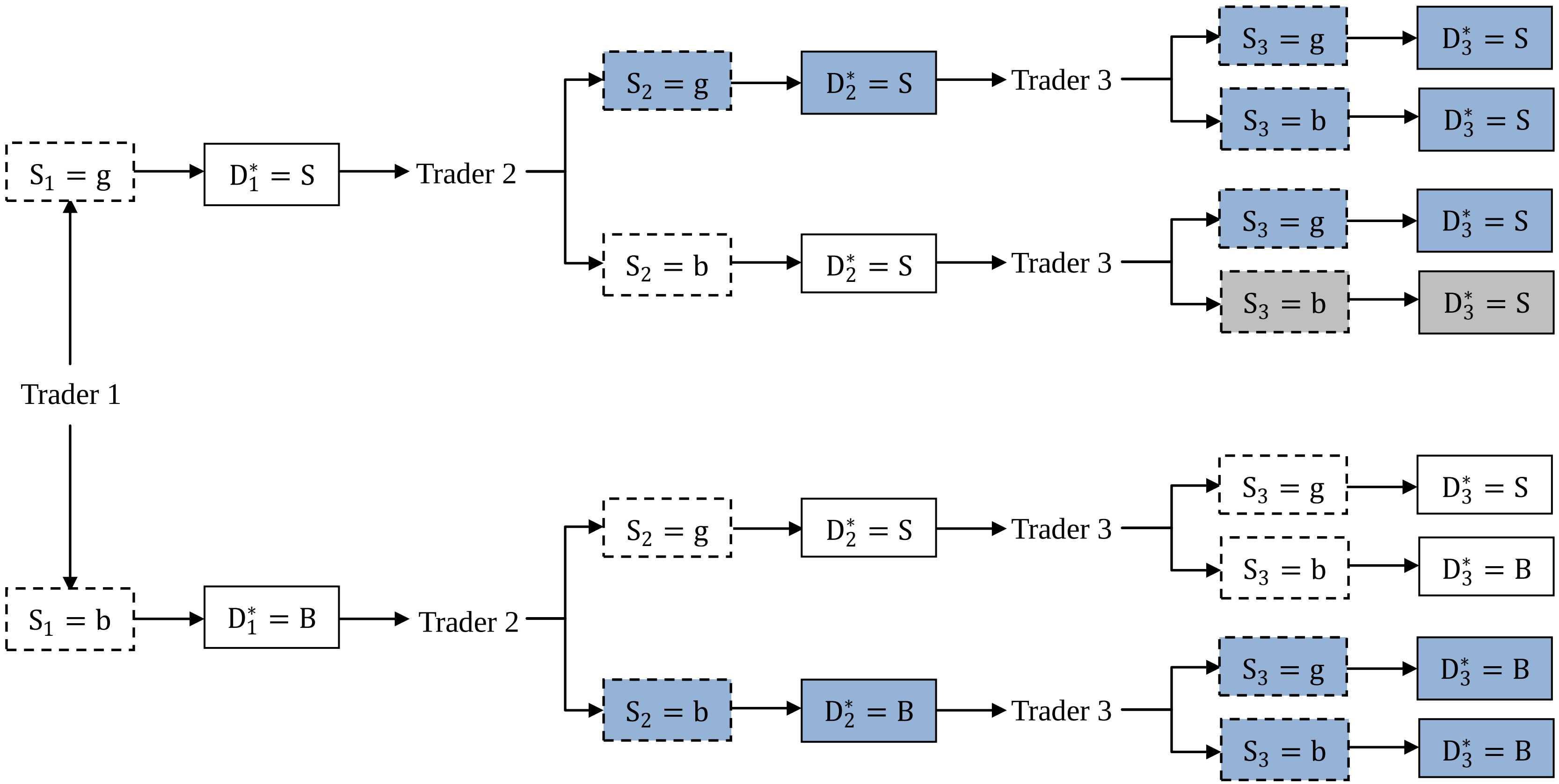
$$\Pr(\sigma_L|g)^{AS} = \frac{Q}{Q+1-P} > \frac{Q}{Q+1-Q} = Q = \Pr(\sigma_L|g)^S; \text{ and } \Pr(\sigma_H|b)^{AS} = \frac{P}{P+1-Q} < \frac{P}{P+1-P} = P = \Pr(\sigma_H|b)^S.$$

[17] Ex ante collateral and ex post monitoring are two common ways to cope with the adverse selection and moral hazard problems of the corporate debtors, however, these two methods are seldom used on sovereign debts.

[18] $\Pr(\sigma_L|S, g) > \Pr(\sigma_L|g) \Leftrightarrow \frac{Q^2}{Q^2+(1-P)^2} > \frac{Q}{Q+1-P} \Leftrightarrow Q^3 + Q^2(1-P) > Q^3 + Q(1-P)^2 \Leftrightarrow Q > 1-P$. This is a basic definition of our model. $\Pr(\sigma_H|B, b) > \Pr(\sigma_H|\, b) \Leftrightarrow \frac{P^2}{P^2+(1-Q)^2} > \frac{P}{P+1-Q} \Leftrightarrow P^3 + P^2(1-Q) > P^3 + P(1-Q)^2 \Leftrightarrow P > 1-Q$. This is also a basic definition of our model.

Figure 1 Information structure and equilibrium strategies

Dotted-line rectangular represents the signal of trader i ($S_i, i = 1,2,3$) of which g means good signal and b means bad signal. While the real-line rectangular indicates the optimal decision of trader i ($D_i^*, i = 1,2,3$) of which B stands for buying CDS and S for selling CDS. The blue-background paths signify a correct cascading effect, whereas the grey-background path exhibits a wrong cascading effect. There is no cascading effect on the rest paths.



However, the second case converges at selling CDS, which is different to all other theoretical papers that point out the second trader just behave randomly, having half a chance to buy and half a chance to sell. This dominating strategy of selling is caused by two reasons. On one hand, the higher reliance of a good sign (selling is regarded as a good signal) makes the trader believe the bad signal might be an accident; on the other hand, selling CDS can gain immediately the premium which could possibly satisfy the liquidity or the leverage needs of the investor.

If the two consecutive actions observed are identical (buying or selling), trader 3 will follow whatever his private signal is. While the previous two actions are adverse, he will behave upon his own private signal. Two consecutive actions always result in a cascading effect of which a wrong cascade (grey-background path in figure 1) should be noticed. The bad signal of trader 2 is hidden behind his selling action, so trader 3 also chooses to sell, which hides again the bad signal. Indeed, more bad signals appear than good signals, which signifies the underlying country probably has a volatile economic growth, but the market believes the opposite. Such underestimation of the default risk only happens when the initial signal is good.[19] On the other hand, two adverse actions let investors behave by making full use of all information by hand, which increases the informational efficiency of the sovereign CDS market. And this occurs only if the initial signal is bad. Therefore, it can be deduced from figure 1 the probabilities of having correct, wrong and no cascading: $\mathrm{Pr}_{\mathrm{Covered}}^{\mathrm{Correct}} = \mathrm{Pr}(\mathrm{b})^2 + \mathrm{Pr}(\mathrm{g})^2 + \mathrm{Pr}(\mathrm{g})^2\,\mathrm{Pr}(\mathrm{b})$; $\mathrm{Pr}_{\mathrm{Covered}}^{\mathrm{Wrong}} = \mathrm{Pr}(\mathrm{g})\,\mathrm{Pr}(\mathrm{b})^2$; $\mathrm{Pr}_{\mathrm{Covered}}^{\mathrm{Non}} = \mathrm{Pr}(\mathrm{b})\,\mathrm{Pr}(\mathrm{g})$.

4.1.3 Microstructure bid-ask evolution

The uncertain signals in our model allow the ex ante optimal decision to be possibly losing ex post, so the bid-ask prices should define a nil ex ante expected profit

[19] Good initial signal can be referred to as a pre-perceived less risky debtor, while bad initial signal is regarded as a pre-impressed risky debtor.

of either buying or selling CDS. The profit of each action is:

$$\begin{cases} \pi(\sigma_H|B) = C(\sigma_H) - A_i^\theta \\ \pi(\sigma_L|B) = C(\sigma_L) - A_i^\theta \\ \pi(\sigma_H|S) = B_i^\theta - C(\sigma_H) \\ \pi(\sigma_L|S) = B_i^\theta - C(\sigma_L) \end{cases} \tag{10}$$

Without the revelation of the true state between t=0 and t=1, the general bid-ask prices are given by lemma 3 under assumption 5.

Assumption 5 In terms of bid-ask spread fixing, the actions between traders are considered as independent as the signal is not observed by the market maker.

Lemma 3 Bid-ask prices are a linear combination of $C(\sigma_H)$ and $C(\sigma_L)$ weighted by the conditional probabilities that each true state realizes updated by the Bayes' rule. All the actions taken before is denoted by θ.

$$A_i^\theta = C(\sigma_H) * \Pr(\sigma_H|\theta, b) + C(\sigma_L) * \Pr(\sigma_L|\theta, b) \tag{11}$$

$$B_i^\theta = C(\sigma_H) * \Pr(\sigma_H|\theta, g) + C(\sigma_L) * \Pr(\sigma_L|\theta, g) \tag{12}$$

Proof: See in appendix.

The bid-ask spread for trader 1 is written beneath as an example:

$$A_1 = C(\overline{\sigma}) + \frac{1}{2}\nabla(\sigma_H - \sigma_L)\frac{P - (1 - Q)}{P + 1 - Q} \tag{13}$$

$$B_1 = C(\overline{\sigma}) - \frac{1}{2}\nabla(\sigma_H - \sigma_L)\frac{Q - (1 - P)}{Q + 1 - P} \tag{14}$$

According to the setting of our model, the initial CDS premium $C(\overline{\sigma})$, the depth of uncertainty $(\sigma_H - \sigma_L)$ and the ∇ all can be considered as constant. The only thing influencing the bid-ask is P and Q which can be regarded as the degree of the signal uncertainty.[20] The less the signals are uncertain, the more the spread widens[21]. This is

[20] The more P (Q) approaches 1, the less the uncertainty of signal is.

[21] $\frac{\partial(A_1-B_1)}{\partial P} = \frac{1}{2}\nabla(\sigma_H - \sigma_L) * \left[\frac{2(1-Q)}{(P+1-Q)^2} + \frac{2Q}{(Q+1-P)^2}\right] > 0$;
$\frac{\partial(A_1-B_1)}{\partial Q} = \frac{1}{2}\nabla(\sigma_H - \sigma_L) * \left[\frac{2P}{(P+1-Q)^2} + \frac{2(1-P)}{(Q+1-P)^2}\right] > 0$.

also consistent with the result of traditional information asymmetry microstructure models that the bid-ask spread is positively correlated with the fraction of insiders.

4.2 The liar equilibrium of naked traders

In our model, we introduce only one naked trader who should make a pair of opposite decisions in the sequential game. Two situations are considered: the naked trader (trader 1) starts the game; and he (trader 2) reacts to a covered trader.

4.2.1 Naked trader as game starter

If the naked trader sells CDS, trader 2 will follow without doubt according to the covered only equilibrium (figure 1). Therefore, he can sell CDS first at the price of B_1 (equation (14)), and buy it back at the price of A_3^{SS}.[22] Contrarily if he buys CDS first at the price of A_1, trader 2 can both buy and sell CDS according to his own signal. Then the naked trader has a probability of $\Pr(b)$ to sell it at the price of B_3^{BB} and $\Pr(g)$ to sell it at B_3^{BS}.

Proposition 3 If the private signal received is:

1) Good. The naked trader sells CDS as covered trader did, but he can certainly buy it at a lower price: $B_1 > A_3^{SS}$.
2) Bad. The naked trader would deviate from the optimal choice as if he was a covered investor, and he realizes a higher profit by selling CDS.

Proof: See in appendix.

This result shows that a naked trader as game starter will never buy CDS first. Even if he receives a bad signal, he would lie through his action, which leaves an

[22] This notation stands for the ask price for trader 3 on condition that trader 1 sells CDS and trader 2 sells CDS. The action notation is consistent in the whole thesis that B represents buying and S for selling.

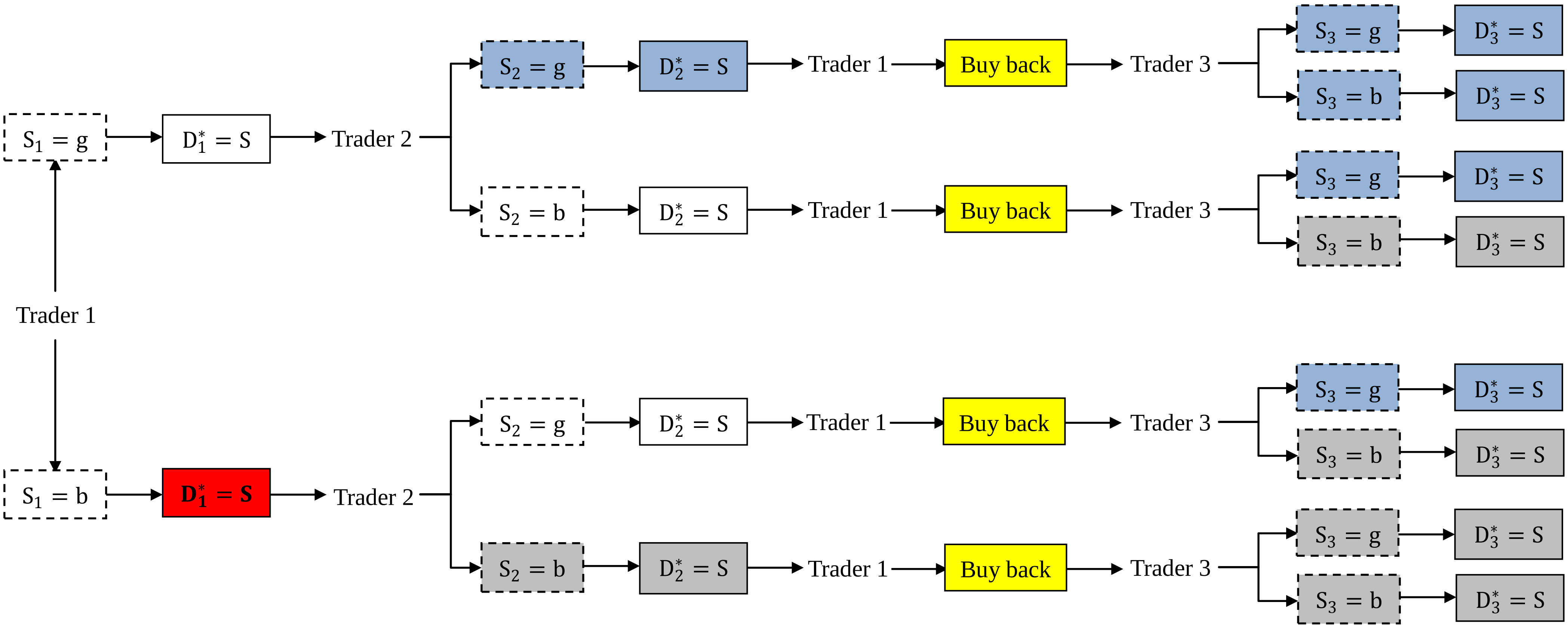
impression to the trader 2 that one good signal has already showed up. This liar equilibrium creates immediately the possibility of a wrong cascading effect if trader 2 receives a bad signal later. So the naked CDS trading makes earlier the wrong cascading effect and thereby sooner the underestimation of the counterparty risk of default. We can further look into the decision making of trader 3 who also evaluates the conditional probability of each true state realizing (lemma 1). Since the second decision of the naked trader (Buy back in figure 2) contains no information concerning the true states, it will be ignored by trader 3 who should respect the covered only equilibrium. Therefore, the game structure and the whole equilibrium of naked starting investors is described in figure 2, from which the probabilities of having correct, wrong and no informational cascade are deduced: $\mathrm{Pr}_{\mathrm{naked\ 1}}^{\mathrm{Correct}} = \mathrm{Pr}(\mathrm{g})^2 + 2\mathrm{Pr}(\mathrm{g})^2\,\mathrm{Pr}(\mathrm{b})$; $\mathrm{Pr}_{\mathrm{naked\ 1}}^{\mathrm{Wrong}} = \mathrm{Pr}(\mathrm{b})^2 + 2\,\mathrm{Pr}(\mathrm{b})^2\,\mathrm{Pr}(\mathrm{g})$; $\mathrm{Pr}_{\mathrm{naked\ 1}}^{\mathrm{Non}} = 0$.

Proposition 4 If the naked trader starts the game without pre-determined sign, at equilibrium, the probabilities of having correct and no cascading effect both decrease, whereas the probability of having wrong informational cascade increases considerably.
Proof: See in appendix.

As is already mentioned in the covered only equilibrium, a wrong cascade indicates a biased estimation of the real default risk concerning the underlying country, while no cascade represents the informational efficiency of the sovereign CDS market. The above result (proposition 4) states that when a naked trader is introduced as a game starter, the biased estimation becomes more biased and the informational efficiency decreases. Therefore, the real default risk is much more underestimated for countries without initial signals.

Figure 2 The equilibrium of the naked starting trading.

Besides the notations in figure 1, yellow-background rectangular represents the second action of the naked trader which contains no information on the true state and which is thereby ignored by trader 3. Trader 3 can neither tell whether the first naked trader has lied nor whether trader 2 has already suffered from the wrong cascade. The red-background rectangular indicates the liar equilibrium strategy. Blue (grey) path signifies a correct (wrong) cascade.



4.2.2 Naked trader as game reactor

On condition that the trader 1 takes covered position, a separating equilibrium will be observed by the naked trader 2. Since the initial good signal results in only one equilibrium action, it is more interesting to study the bad initial signal. If he buys CDS first at the price of A_2^S, trader 3 will always buy CDS, according to the covered only equilibrium; then he can sell it back at the price of B_4^{BBB}.[23] If he sells CDS first at the price of B_2^S, trader 3 has the probability of bad signal occurring to buy CDS, which enables the naked trader to buy it back at the price of A_4^{BSB}; and the probability of good signal occurring to sell CDS, which allows the naked trader to buy it back at the price of A_4^{BSS}. The whole naked reacting equilibrium is presented in figure 3.

Proposition 5 Known that the covered trader 1 bought CDS, the naked trader will always buy CDS, whatever message he receives.

$$B_4^{BBB} - A_2^S > \Pr(b) * \left(B_2^S - A_4^{BSB}\right) + \Pr(g) * \left(B_2^S - A_4^{BSS}\right) \qquad (15)$$

Proof: See in appendix.

This shows the naked trader 2 will deviate from the covered only equilibrium when he receives a good signal. However, this deviation needs to be justified, because unfortunately the profit of this optimal deviation is always negative $B_4^{BBB} - A_2^S < 0$.[24] This is not what the naked speculator pursues, so he must hold the CDS longer to wait for more investors joining the consecutive buying cascade, which increases the premium until $B_i^{B^{i-1}} - A_2^S > 0$.[25] Although the longer the naked trader holds the CDS, the more investors follow the wrong informational cascade, and the more he could gain

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[23] Unlike traditional option microstructure theoretic papers, the transaction cost is taken into account both at buying and selling in this dissertation.

[24] $B_4^{BBB} - A_2^S < 0 \Leftrightarrow \frac{1}{2}\nabla(\sigma_H - \sigma_L)\left[\frac{P^3(1-P)-(1-Q)^3Q}{P^3(1-P)+(1-Q)^3Q} - \frac{P^2-(1-Q)^2}{P^2+(1-Q)^2}\right] < 0 \Leftrightarrow \frac{P^3(1-P)-(1-Q)^3Q}{P^3(1-P)+(1-Q)^3Q} < \frac{P^2-(1-Q)^2}{P^2+(1-Q)^2} \Leftrightarrow$ $P^3(1-P)(1-Q)^2 < P^2(1-Q)^3Q \Leftrightarrow P(1-P) < (1-Q)Q$. The last inequality has already been proved in proposition 3.

[25] As the naked trader plays in the second place, $i-3$ traders follow the wrong cascade created by the liar equilibrium. $B_i^{B^{i-1}}$ stands for the bid price for the i^{th} trader known that the previous $i-1$ actions are all buying.

by selling it as a result of $B_i^{B^{i-1}} < B_{i+1}^{B^i}, \forall i$; also the more the unexpected CDS price changes are likely to take place, and the more potential losses are to be realized. Therefore, a tradeoff must be made by naked traders. The impact of trading naked CDS as a game reactor is presented below.

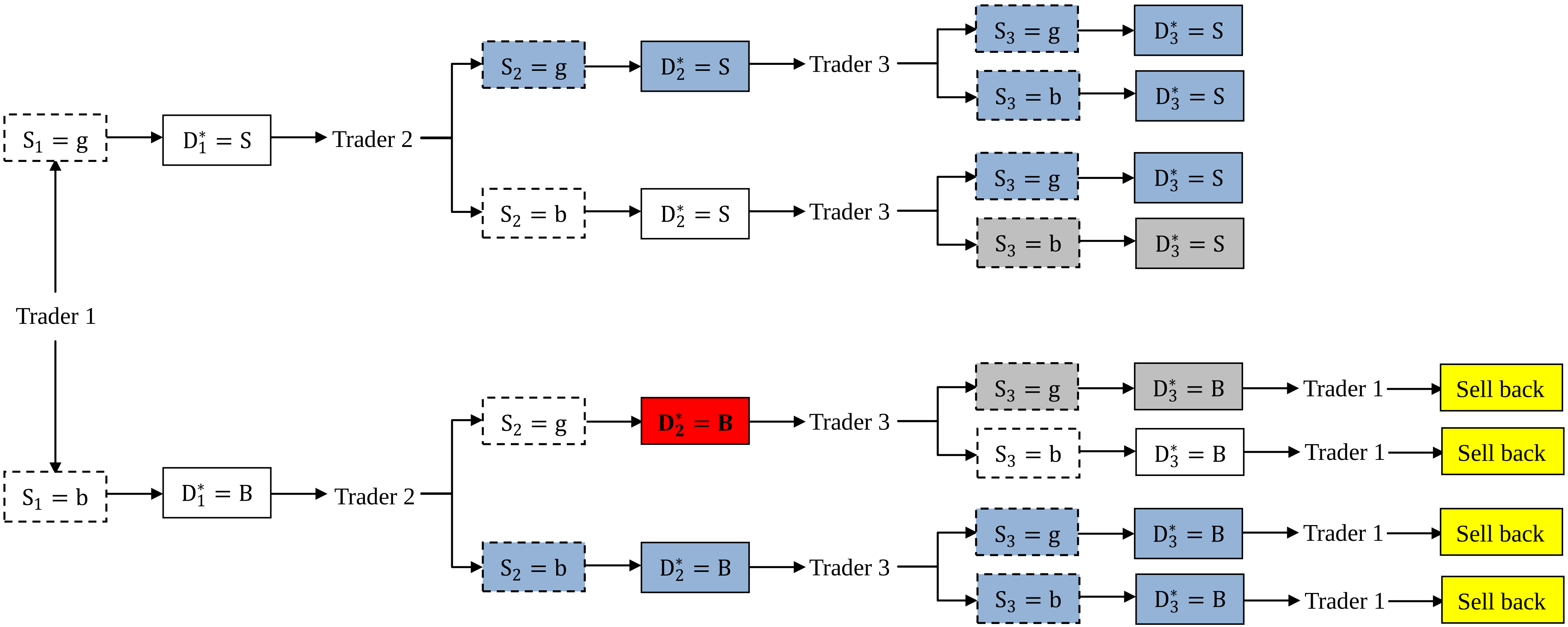
Proposition 6 The possibility of having no cascading effects reduced to compensate that of having wrong cascading effect, while the probability of having correct cascade remains unchanged.
Proof: See in appendix.

Under the present equilibrium, the estimation of the real default risk is also more biased, and more precisely, the default risk of a country with initial bad signal is overestimated compared to the covered only equilibrium. In addition, the speculators themselves are also exposed to more market risks, for a longer holding of the speculating asset, namely the sovereign CDS in our model.

To sum up, as pure speculator, the naked sovereign trader, no matter he starts the game or reacts in the second place, realizes the liar equilibrium where the optimal decisions are contradictory to the signals received. Because cascading effect is likely to form even only with covered hedgers, the naked trader manages to mislead the action of the covered trader by deceiving him into a wrong cascade created by the liar equilibrium. Such manipulations enable the naked trader offsetting his previous opposite speculating position with the maximum profit. Consequently, the wrong cascading effect can be amplified and the CDS market estimates less accurately the counterparty risks of the sovereign debts. Especially, naked starting trading biases more the market anticipation.

Figure 3 The equilibrium of the naked reacting trading.

The notations are completely identical to those in figure 2. The probability of having correct (or wrong) informational cascade can be intuitively obtained by the number of blue (or grey) rectangular.



5. Discussions and further econometric issues

5.1 The guilt of naked sovereign CDS trading

In this subsection, we would like to make a marginal contribution to the existing debate on whether naked CDS trading should be banned, with the findings of our model. Let us refer to the economy of our model in equation (2) and (3), it is noticed that the real probability of default is an increasing function of the repayment success threshold[26] which is also logically an increasing function of the interest rate (R_i) demanded by credit lenders. If the credit lender is on a perfectly-competitive market, the interest rate should be fixed in the following way:[27]

$$(1 + R_i) * \left(1 - \widehat{Def}\right) + k * \widehat{Def} = 1 + \mu \qquad (16)$$

The credit lenders should have used the real probability of default to determine the interest rate, unfortunately they do not know it, and thus they could only use the market anticipation as a reference to calculate the interest rate.

$$R_i = \frac{\mu - k * \widehat{Def} + \widehat{Def}}{1 - \widehat{Def}} \qquad (17)$$

Therefore, the interest rate is also an increasing function of the estimated probability of default,[28] which means the real probability of default is an increasing function of the estimated probability. A self-fulfilling equilibrium can potentially be achieved that the more the market anticipates a default, the more the default is likely to take place, and thereby the more the market anticipates again a default; vice versa. As our model suggests that naked traders practice a deviating liar equilibrium, which

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26 $\frac{\partial Def}{\partial R} = \frac{\partial N\left(\frac{\ln\frac{R}{V(0)} - \left(\mu - \frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}\right)}{\partial R} > 0$

27 We have already defined in our model μ as the risk-free rate and k the recovery rate in case of credit event; $\widehat{Def}$ is the estimated probability of default from the issuing of the debt to the expiration.

28 $\frac{\partial R_i}{\partial \widehat{Def}} = \frac{\partial \frac{\mu - k * \widehat{Def} + \widehat{Def}}{1 - \widehat{Def}}}{\partial \widehat{Def}} = \frac{\mu + 1 - k}{\left(1 - \widehat{Def}\right)^2} > 0$

results in an earlier start of a more possible wrong cascading effect and hence a more biased estimation of the real default risk. The biased estimation will be then amplified by the self-fulfilling equilibrium, which eventually makes the loan repayment much easier for countries without initial sign but much harder for those with initial bad signs. This agrees with the concern of many economists on the default of sovereign debts.

However, the market behavior only influences the target threshold for the country debtor, yet never the distribution of the country's economic growth whose sole variable is the volatility σ, a factor independent from the CDS market. In the mathematic sense of our model, the country should decrease as much as possible σ to realize a stable economic growth whose distribution has a higher mean and a thinner tail.

Take the example of Greece; trading Greek sovereign CDS has certainly negative impacts on its debt repayment, for instance, the yield demanded by credit lenders increases as CDS spread widens, rating agencies also degrade Greek sovereign debt rating, and etc. But the Greek crisis is caused fundamentally by its economic malfunction. For example, the high level of debt to GDP ratio cannot be cut down after Greece joining the E.U, because the currency depreciation, which contributed a lot to Greece's fast economic growth from the late 1990s to early 2000s, no longer works. This harms greatly two largest Greek industries: tourism and shipping that are regarded as the exportation of Greece, and then the whole Greek economy. In this particular case, it is the decision to join the E.U rather than the naked sovereign speculating that should be questioned.

Generally speaking, naked CDS trading is only responsible of creating biased estimation of the default risks, which makes more difficult the repayment of risky borrowers but easier for less risky countries. However, it cannot force a country to default or contrarily save a country from default, because whether the default takes place is determined by the country's economic development and strength.

5.2 Further econometric issues

There are several assumptions in our model on which empirical tests could be conducted so as to study whether these assumptions stand on real market. In our model, we assumed in lemma 1 the absence of arbitrage opportunity on CDS market, so it is interesting to know whether the sovereign CDS is correctly priced as well as the determinants of the spread (by a multi-factor model). Besides, as a financial asset, the sovereign CDS follows probably the famous random walk, but does a co-integration relation exist with other variables, which makes possible the prediction of future price. Moreover, case study can be conducted in order to find out how the sovereign market responds to public information disclosure. These issues will be further studied by the author of this Master's thesis in his doctoral thesis.

6. Conclusion

To conclude, our model establishes a sequential sovereign CDS trading model based on a dynamic microstructure bid-ask spread within an informational cascade framework. All the investors are classified as the covered trader who is concerned about the global payoff of both his debt and the CDS, and the naked trader who only cares if he could buy CDS at a lower price and resell it at a higher price.

The main findings of our model suggest that at the absence of naked speculators, covered hedgers buy (sell) CDS, if the underlying economic growth is more likely to be volatile (stable) than stable (volatile). The cascading effect will form, of which a wrong cascade underestimates the probability of default of an initially well impressed country. When naked traders are introduced, they realize the liar equilibrium where the optimal decisions are contradictory to the signals received. This enables naked traders to mislead the action of the following covered traders by deceiving them into a wrong cascade created by the liar equilibrium, which contrarily results in an arbitrage of the naked traders. Therefore, the wrong cascading effect forms earlier and more possibly at the presence of naked traders, hence the default risk of the underlying country is less accurately estimated.

However, the guilt of naked CDS trading stops here, because it can never force a country to default or save a country from the failure. A country should have a strong and stable economic growth that enables him to fully repay the loan, which is not determined by the market anticipation of CDS traders, but by the development of its financial system, the monetary and fiscal policies, the technology innovation, the human resources, and etc.

Further development can still be added to our theoretical model, for instance, more possible situations of the signal uncertainties can be discussed, whereas in our model the uncertainty degrees of both good and bad signal are alike. Besides, the presence of more than one naked speculator in one sequential game can also be introduced. Moreover, we plan to realize the mentioned econometric issues in future studies.

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Appendix

Proposition 1

The derivative can be decomposed as follows:

$$\frac{\partial \mathrm{Def}}{\partial \sigma} = \frac{\partial \mathrm{Def}}{\partial(-\mathrm{d}_2)} * \frac{\partial(-\mathrm{d}_2)}{\partial \sigma} \tag{18}$$

Since the Cumulative Distribution Function (CDF) of a standard normal distribution is increasing, it is evident that

$$\frac{\partial \mathrm{Def}}{\partial(-\mathrm{d}_2)} > 0 \tag{19}$$

In addition,

$$\frac{\partial(-\mathrm{d}_2)}{\partial \sigma} = \frac{\ln\frac{\mathrm{V}(0)}{\mathrm{R}} + \left(\mu + \frac{1}{2}\sigma^2\right)\mathrm{t}}{\sigma^2 \mathrm{t}} = \frac{\mathrm{d}_1}{\sigma\sqrt{\mathrm{t}}} \tag{20}$$

As mentioned that a CDS is similar to a put option, which should not be deep in-the-money when it is issued; in this situation d_1 should be positive. Therefore:

$$\frac{\partial(-\mathrm{d}_2)}{\partial \sigma} > 0; \text{ and } \frac{\partial \mathrm{Def}}{\partial \sigma} > 0$$

Lemma 1

According to equation (6) and the risk neutral assumption, we can express the expected utility of each action as follows:

$$\begin{cases} \mathrm{Eu}(\mathrm{B}|\Omega) = \pi(\mathrm{B}|\mathrm{Def}) * \Pr(\mathrm{Def}|\sigma_\mathrm{H}) * \Pr(\sigma_\mathrm{H}|\Omega) + \pi(\mathrm{B}|\mathrm{Def}) * \Pr(\mathrm{Def}|\sigma_\mathrm{L}) * \Pr(\sigma_\mathrm{L}|\Omega) + \\ \pi(\mathrm{B}|\text{No Def}) * \Pr(\text{No Def}|\sigma_\mathrm{H}) * \Pr(\sigma_\mathrm{H}|\Omega) + \pi(\mathrm{B}|\text{No Def}) * \Pr(\text{No Def}|\sigma_\mathrm{L}) * \Pr(\sigma_\mathrm{L}|\Omega) \\ \\ \mathrm{Eu}(\mathrm{S}|\Omega) = \pi(\mathrm{S}|\mathrm{Def}) * \Pr(\mathrm{Def}|\sigma_\mathrm{H}) * \Pr(\sigma_\mathrm{H}|\Omega) + \pi(\mathrm{S}|\mathrm{Def}) * \Pr(\mathrm{Def}|\sigma_\mathrm{L}) * \Pr(\sigma_\mathrm{L}|\Omega) + \\ \pi(\mathrm{S}|\text{No Def}) * \Pr(\text{No Def}|\sigma_\mathrm{H}) * \Pr(\sigma_\mathrm{H}|\Omega) + \pi(\mathrm{S}|\text{No Def}) * \Pr(\text{No Def}|\sigma_\mathrm{L}) * \Pr(\sigma_\mathrm{L}|\Omega) \end{cases}$$

Having replaced the payoffs and the conditional probabilities from figure 2, to judge if $\mathrm{Eu}(\mathrm{A}|\Omega) \lesseqgtr \mathrm{Eu}(\mathrm{V}|\Omega)$ is equivalent to evaluate if:

$$\left(K - A_i^\theta\right) * Def_H * Pr(\sigma_H|\Omega) + \left(K - A_i^\theta\right) * Def_L * Pr(\sigma_L|\Omega) - A_i^\theta * (1 - Def_H) * Pr(\sigma_H|\Omega) - A_i^\theta * (1 - Def_L) * Pr(\sigma_L|\Omega) \lesseqgtr \left(B_i^\theta - K\right) * Def_H * Pr(\sigma_H|\Omega) + \left(B_i^\theta - K\right) * Def_L * Pr(\sigma_L|\Omega) + B_i^\theta * (1 - Def_H) * Pr(\sigma_H|\Omega) + B_i^\theta * (1 - Def_L) * Pr(\sigma_L|\Omega) \tag{21}$$

By rearranging terms we have:

$$\left(2K - A_i^\theta - B_i^\theta\right) * \left(Def_H * Pr(\sigma_H|\Omega) + Def_L * Pr(\sigma_L|\Omega)\right) \lesseqgtr \left(A_i^\theta + B_i^\theta\right) * [(1 - Def_H) * Pr(\sigma_H|\Omega) + (1 - Def_L) * Pr(\sigma_L|\Omega)] \tag{22}$$

As a matter of fact only two states of nature are possible in our model, by replacing $Pr(\sigma_H|\Omega) = 1 - Pr(\sigma_L|\Omega)$ into the above relation and simplifying terms:

$$Pr(\sigma_H|\Omega) \lesseqgtr \frac{\frac{A_i^\theta + B_i^\theta}{2} / K - Def_L}{Def_H - Def_L} \tag{23}$$

Assuming the absence of arbitrage opportunity on CDS market, we believe that the mid-spread should equal the compensation multiplied by the average probability of default. So we have:

$$\frac{A_i^\theta + B_i^\theta}{2} = K * \frac{Def_H + Def_L}{2} \iff \frac{\frac{A_i^\theta + B_i^\theta}{2} / K - Def_L}{Def_H - Def_L} = \frac{1}{2} \gtreqless Pr(\sigma_H|\Omega) \tag{24}$$

To conclude, $Eu(A|\Omega) \lesseqgtr Eu(V|\Omega)$ and $Pr(B|\Omega) \lesseqgtr 1/2$ or $Pr(G|\Omega) \gtreqless 1/2$ are equivalent.

Lemma 2

As we have already deduced from the first agent's separate equilibrium strategy that buying represents a bad signal and selling indicates a good signal; so we have $Pr(B\,|\lambda) = Pr(b\,|\lambda)$ and $Pr(S\,|\lambda) = Pr(g\,|\lambda)$. Under the assumption of independence between previous actions and the signal, the conditional probability can be rewritten as follows:

$$\Pr\left(\sigma_H \middle| \underbrace{B,B,\dots,B}_{j},\underbrace{S,S,\dots,S}_{n-1-j},b\right) = \frac{\Pr\left(\underbrace{B,B,\dots,B}_{j},\underbrace{S,S,\dots,S}_{n-1-j},b \middle| \sigma_H\right) * \Pr(\sigma_H)}{\sum_{m=H,L} \Pr\left(\underbrace{B,B,\dots,B}_{j},\underbrace{S,S,\dots,S}_{n-1-j},b \middle| \sigma_m\right) * \Pr(\sigma_m)}$$

$$= \frac{\Pr(B\,|\sigma_H)^j * \Pr(S\,|\sigma_H)^{n-1-j} * \Pr(b\,|\sigma_H)}{\sum_{m=H,L} \Pr(B\,|\sigma_m)^j * \Pr(S\,|\sigma_m)^{n-1-j} * \Pr(b\,|\sigma_m)}$$

$$= \frac{\Pr(b\,|\sigma_H)^{j+1} * \Pr(g\,|\sigma_H)^{n-1-j}}{\sum_{m=H,L} \Pr(b\,|\sigma_m)^{j+1} * \Pr(g\,|\sigma_m)^{n-1-j}}$$

$$= \frac{P^{j+1} * (1-P)^{n-1-j}}{P^{j+1} * (1-P)^{n-1-j} + (1-Q)^{j+1} * Q^{n-j-1}} \qquad (25)$$

$$\Pr\left(\sigma_L \middle| \underbrace{B,B,\dots,B}_{j},\underbrace{S,S,\dots,S}_{n-1-j},g\right) = \frac{\Pr\left(\underbrace{B,B,\dots,B}_{j},\underbrace{S,S,\dots,S}_{n-1-j},g \middle| \sigma_L\right) * \Pr(\sigma_L)}{\sum_{m=H,L} \Pr\left(\underbrace{B,B,\dots,B}_{j},\underbrace{S,S,\dots,S}_{n-1-j},g \middle| \sigma_m\right) * \Pr(\sigma_m)}$$

$$= \frac{\Pr(b\,|\sigma_L)^j * \Pr(g\,|\sigma_L)^{n-1-j} * \Pr(g\,|\sigma_L)}{\sum_{m=H,L} \Pr(b\,|\sigma_m)^j * \Pr(g\,|\sigma_m)^{n-1-j} * \Pr(g\,|\sigma_m)}$$

$$= \frac{\Pr(b\,|\sigma_L)^j * \Pr(g\,|\sigma_L)^{n-j}}{\sum_{m=H,L} \Pr(b\,|\sigma_m)^j * \Pr(g\,|\sigma_m)^{n-j}}$$

$$= \frac{(1-Q)^j * Q^{n-j}}{(1-Q)^j * Q^{n-j} + P^j * (1-P)^{n-j}} \qquad (26)$$

Proposition 2

The proof of the covered equilibrium is made by first determining the conditional probabilities through lemma 2, and then by comparing the probability to one half as lemma 1 suggests. Model definition is used that $P > Q > \frac{1}{2} > 1 - Q > 1 - P$.

1) Trader 1 has a separating equilibrium.

$$\Pr(\sigma_L|S_1 = g) = \frac{Q}{Q + 1 - P} > \frac{Q}{Q + Q} = \frac{1}{2}; \qquad (27)$$

$$\Pr(\sigma_H|S_1 = b) = \frac{P}{P + 1 - Q} > \frac{P}{P + P} = \frac{1}{2}. \qquad (28)$$

2.1) If the signal of trader 2 corresponds to the action of trader 1.

$$\Pr(\lambda = \sigma_H | D_1 = B, S_2 = b) = \frac{P^2}{P^2 + (1-Q)^2} > \frac{P^2}{P^2 + P^2} = \frac{1}{2}; \tag{29}$$

$$\Pr(\lambda = \sigma_L | D_1 = S, S_2 = g) = \frac{Q^2}{Q^2 + (1-P)^2} > \frac{Q^2}{Q^2 + Q^2} = \frac{1}{2}. \tag{30}$$

2.2) If the signal of trader 2 contradicts to the action of trader 1.

Define $f(x) = x(1-x)$, so $f'(x) = 1 - 2x$. If $x \in (0.5, 1)$, $f'(x) < 0$. That is to say

$$f(Q) > f(P): (1-Q)Q > P(1-P) \tag{31}$$

According to the result of equation (31), we can then determine:

$$\Pr(\lambda = \sigma_H | D_1 = S, S_2 = b) = \frac{(1-P)P}{(1-P)P + Q(1-Q)} < \frac{\frac{1}{2} * (1-P)P + \frac{1}{2} * Q(1-Q)}{(1-P)P + Q(1-Q)} = \frac{1}{2}. \tag{32}$$

$$\Pr(\lambda = \sigma_L | D_1 = B, S_2 = g) = \frac{(1-Q)Q}{(1-Q)Q + P(1-P)} > \frac{\frac{1}{2} * Q(1-Q) + \frac{1}{2} * (1-P)P}{(1-P)P + Q(1-Q)} = \frac{1}{2}. \tag{33}$$

3.1) Consecutive buying and selling are respectively considered.

(1) $D_1 = B, D_2 = B$

$$\Pr(\lambda = \sigma_H | D_1 = B, D_2 = B, S_3 = b) = \frac{P^3}{P^3 + (1-Q)^3} > \frac{P^3}{P^3 + P^3} = \frac{1}{2}. \tag{34}$$

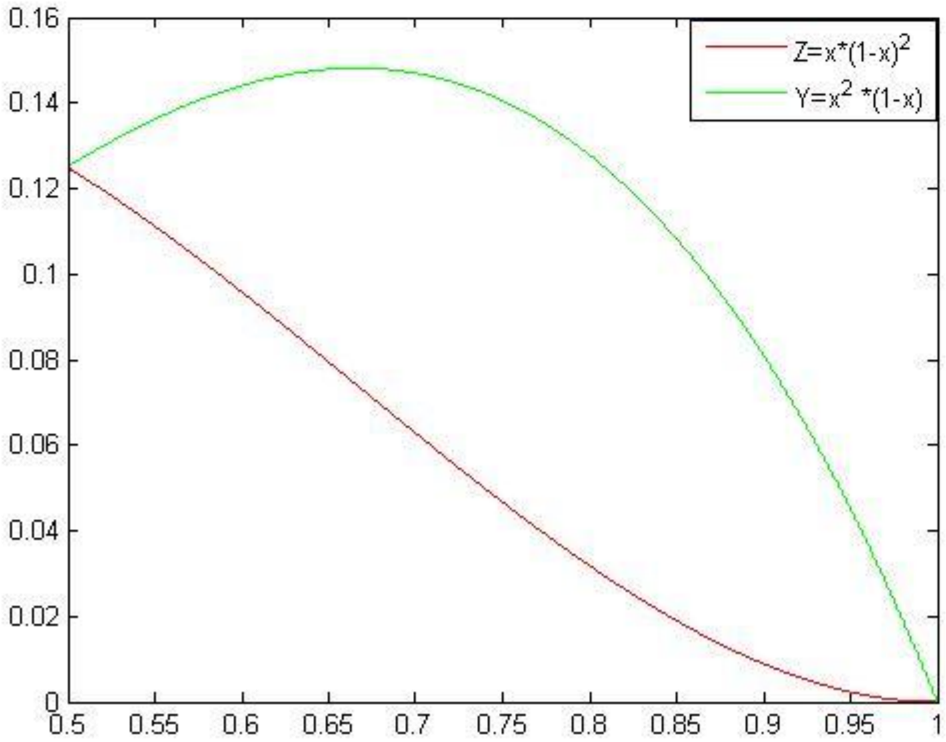
$$\Pr(\lambda = \sigma_H | D_1 = B, D_2 = B, S_3 = g) = \frac{P^2(1-P)}{P^2(1-P) + (1-Q)^2 Q} = \frac{1}{2} + \frac{1}{2} * \frac{P^2(1-P) - (1-Q)^2 Q}{P^2(1-P) + (1-Q)^2 Q}; \tag{35}$$

Let us define two functions $Y = x^2(1-x)$ and $Z = x(1-x)^2$ with $x \in (0.5,1)$, which are presented in the figure 4[29]. It can be easily noticed that the Y curve is always placed in the upper area of the Z curve, so for $P > Q$, it is always true that $Y(Q) > Z(P)$, however it is possible that $Y(P) < Z(Q)$ if Q is around 0.5 and P is around 1. As

[29] Figure 4, 5, 6 are all generated by the software MATLAB.

assumption 2 suggests that $P - Q$ be small enough for the signal uncertainty should not be too big.

Figure 4 Function comparison



So we have:

$$\begin{cases} Y(P) = P^2(1-P) > Z(Q) = Q(1-Q)^2 \\ Y(Q) = Q^2(1-Q) > Z(P) = P(1-P)^2 \end{cases} \qquad \forall\, P, Q \in (0.5,1) \tag{36}$$

Equation (36) implies that: $\Pr(\lambda = \sigma_H | D_1 = B, D_2 = B, S_3 = g) > \frac{1}{2}$

In general, no matter what signal A3 receives, he always chooses to buy CDS.

(2) $D_1 = S, D_2 = S$

$$\Pr(\lambda = \sigma_L | D_1 = S, D_2 = S, S_3 = g) = \frac{Q^3}{Q^3 + (1-P)^3} > \frac{Q^3}{Q^3 + Q^3} = \frac{1}{2} \tag{37}$$

By using equation (36), we can determine that

$$\Pr(\lambda = \sigma_L | D_1 = S, D_2 = S, S_3 = b) = \frac{Q^2(1-Q)}{Q^2(1-Q) + (1-P)^2 P}$$

$$> \frac{\frac{1}{2} * Q^2(1-Q) + \frac{1}{2} * (1-P)^2 P}{Q^2(1-Q) + (1-P)^2 P} = \frac{1}{2}. \tag{38}$$

3.2) Two adverse actions observed can only be “B,S”, as selling at first denies buying next. The result of equation (36) is still used.

$$\Pr(\lambda = \sigma_H | D_1 = B, D_2 = S, S_3 = g) = \frac{P(1-P)^2}{P(1-P)^2 + (1-Q)Q^2}$$

$$< \frac{\frac{1}{2} * P(1-P)^2 + \frac{1}{2} * (1-Q)Q^2}{P(1-P)^2 + (1-Q)Q^2} = \frac{1}{2}; \tag{39}$$

$$\Pr(\lambda = \sigma_H | D_1 = B, D_2 = S, S_3 = b) = \frac{P^2(1-P)}{P^2(1-P) + (1-Q)^2 Q}$$

$$> \frac{\frac{1}{2} * P^2(1-P) + \frac{1}{2} * (1-Q)^2 Q}{P^2(1-P) + (1-Q)^2 Q} = \frac{1}{2}. \tag{40}$$

Lemma 3

The market maker should fix the bid and ask prices so that the ex ante expected profit of each action equals zero.

$$\begin{cases} \pi(\sigma_H|B) * \Pr(\sigma_H|B) + \pi(\sigma_L|B) * \Pr(\sigma_L|B) = 0 \\ \pi(\sigma_H|S) * \Pr(\sigma_H|S) + \pi(\sigma_L|S) * \Pr(\sigma_L|S) = 0 \end{cases} \tag{41}$$

By replacing equation (10) in (40), we have:

$$\begin{cases} \left(C(\sigma_H) - A_i^\theta\right) * \Pr^\theta(\sigma_H|B) + \left(C(\sigma_L) - A_i^\theta\right) * \Pr^\theta(\sigma_L|B) = 0 \\ \left(B_i^\theta - C(\sigma_H)\right) * \Pr^\theta(\sigma_H|S) + \left(B_i^\theta - C(\sigma_L)\right) * \Pr^\theta(\sigma_L|S) = 0 \end{cases} \tag{42}$$

With $\Pr^\theta$ the conditional probability after the whole set of actions (θ) has realized.

By rearranging terms, we have

$$\begin{cases} A_i^\theta = C(\sigma_H) * \Pr^\theta(\sigma_H|B) + C(\sigma_L) * \Pr^\theta(\sigma_L|B) \\ B_i^\theta = C(\sigma_H) * \Pr^\theta(\sigma_H|S) + C(\sigma_L) * \Pr^\theta(\sigma_L|S) \end{cases} \tag{43}$$

The assumption 5 signifies that even under mixed equilibrium action, the market maker still need to fix both bid and ask prices, by assuming the action of a certain trader is led by his own signal. That is:

$$\begin{cases} \Pr^{\theta}(\sigma_H|B) = \Pr(\sigma_H|\theta,b), \Pr^{\theta}(\sigma_L|B) = \Pr(\sigma_L|\theta,b); \\ \Pr^{\theta}(\sigma_H|S) = \Pr(\sigma_H|\theta,g), \Pr^{\theta}(\sigma_L|S) = \Pr(\sigma_L|\theta,g); \end{cases} \tag{44}$$

So lemma 3 is verified by replacing equation (44) in (43)

Proposition 3

The proof is made by backward induction.

1) It is necessary to calculate A_3^{SS} first according to lemma 2 and 3, since B_1 is already given by (14).

$$\begin{aligned} A_3^{SS} &= C(\sigma_H) * \Pr(\sigma_H|S,S,b) + C(\sigma_L) * \Pr(\sigma_L|S,S,b) \\ &= C(\sigma_H) * \frac{(1-P)^2P}{Q^2(1-Q)+(1-P)^2P} + C(\sigma_L) * \frac{Q^2(1-Q)}{Q^2(1-Q)+(1-P)^2P} \\ &= C(\overline{\sigma}) - \frac{1}{2}\nabla(\sigma_H-\sigma_L)\frac{Q^2(1-Q)-(1-P)^2P}{Q^2(1-Q)+(1-P)^2P} \end{aligned} \tag{45}$$

So the profit of selling first and buying back is as follows:

$$B_1 - A_3^{SS} = \frac{1}{2}\nabla(\sigma_H-\sigma_L)\left[\frac{Q^2(1-Q)-P(1-P)^2}{Q^2(1-Q)+P(1-P)^2} - \frac{Q-(1-P)}{Q+1-P}\right] \tag{46}$$

As $\frac{1}{2}\nabla(\sigma_H-\sigma_L)$ is a positive constant, to prove $B_1 > A_3^{SS}$ is to verify:

$$\frac{Q^2(1-Q)-P(1-P)^2}{Q^2(1-Q)+P(1-P)^2} - \frac{Q-(1-P)}{Q+1-P} > 0 \tag{47}$$

We can rewrite inequation (47) as follows:

$$\begin{aligned} Q^2(1-Q) + 2Q^2(1-Q)(1-P) - P(1-P)^3 &> Q^2(1-Q) \\ &+ 2QP(1-P)^2 - P(1-P)^3 \end{aligned} \tag{48}$$

By simplifying terms we have

$$(1-Q)Q > P(1-P)$$

And this is what we have already proved by equation (31).

2) The profit of deviating is that of the arbitrage selling strategy $B_1 - A_3^{SS} > 0$. On the other hand, the profit of buying first is that $\Pr(b) * (B_3^{BB} - A_1) + \Pr(g) * \left(B_3^{BS} - A_1\right)$. If to deviate is more profitable, then

$$B_1 - A_3^{SS} > \Pr(b) * (B_3^{BB} - A_1) + \Pr(g) * \left(B_3^{BS} - A_1\right) \tag{49}$$

As $B_3^{BS} - A_1 < 0$ and $\Pr(b) < 1$. So it is sufficient to prove that

$$B_1 - A_3^{SS} > B_3^{BB} - A_1 \tag{50}$$

As A_1 is given by equation (13), we should calculate B_3^{BB} by lemma 2 and 3.

$$\begin{aligned} B_3^{BB} &= C(\sigma_H) * \Pr(\sigma_H|B,B,g) + C(\sigma_L) * \Pr(\sigma_L|B,B,g) \\ &= C(\sigma_H) * \frac{P^2(1-P)}{P^2(1-P)+(1-Q)^2Q} + C(\sigma_L) * \frac{(1-Q)^2Q}{P^2(1-P)+(1-Q)^2Q} \\ &= C(\overline{\sigma}) + \frac{1}{2}\nabla(\sigma_H - \sigma_L)\frac{P^2(1-P)-(1-Q)^2Q}{P^2(1-P)+(1-Q)^2Q} \end{aligned} \tag{51}$$

By replacing the value of each price in equation (50), we have

$$\frac{Q^2(1-Q)-P(1-P)^2}{Q^2(1-Q)+P(1-P)^2} - \frac{Q-(1-P)}{Q+1-P} > \frac{P^2(1-P)-Q(1-Q)^2}{P^2(1-P)+Q(1-Q)^2} - \frac{P-(1-Q)}{P+1-Q} \tag{52}$$

By multiplying the least common multiple of the denominator, we have

$$\begin{aligned} &P^3Q^2(1-P)^2(1-Q) + PQ^3(1-Q)^3(1-P) + Q^3(1-Q)^4(1-P) - P^4Q(1-P)^3 \\ &\quad - P^3Q(1-P)^3(1-Q) - PQ^2(1-Q)^3(1-P)^2 \\ &\quad > P^2Q^3(1-P)(1-Q)^2 + P^3Q(1-P)^3(1-Q) + P^3(1-P)^4(1-Q) \\ &\quad - PQ^4(1-Q)^3 - PQ^3(1-Q)^3(1-P) - P^2Q(1-P)^3(1-Q)^2 \end{aligned} \tag{53}$$

By regrouping terms we have

$$\begin{aligned} &[PQ + (1-P)(1-Q)][Q^3(1-Q)^3 - P^3(1-P)^3] \\ &+2PQ(1-P)(1-Q)[Q^2(1-Q)^2 - P^2(1-P)^2] \\ &-[P^2Q^2(1-P)(1-Q) + PQ(1-P)^2(1-Q)^2][Q(1-Q) - P(1-P)] > 0 \end{aligned} \tag{54}$$

By isolating the positive common multiplier $[Q(1-Q) - P(1-P)]$ we have

$$\begin{aligned} &[Q(1-Q) - P(1-P)]\{[PQ + (1-P)(1-Q)][Q^2(1-Q)^2 + P^2(1-P)^2] \\ &\quad + 2PQ(1-P)(1-Q)[Q(1-Q) + P(1-P)]\} > 0 \end{aligned} \tag{55}$$

Finally each element on the left side is positive so that this inequation is satisfied.

Proposition 4

This proposition suggests the three following relationships: $\Pr_{\text{naked 1}}^{\text{Wro ng}} > \Pr_{\text{Covered}}^{\text{Wrong}}$; $\Pr_{\text{naked 1}}^{\text{Non}} < \Pr_{\text{Covered}}^{\text{Non}}$ and $\Pr_{\text{naked 1}}^{\text{Correct}} < \Pr_{\text{Covered}}^{\text{Correct}}$, which can all be verified by backward induction.

1) $\mathrm{Pr}_{\text{naked 1}}^{\text{Correct}} < \mathrm{Pr}_{\text{Covered}}^{\text{Correct}}$ is exactly the same to:

$$\mathrm{Pr(g)}^2 + 2\mathrm{Pr(g)}^2\,\mathrm{Pr(b)} < \mathrm{Pr(b)}^2 + \mathrm{Pr(g)}^2 + \mathrm{Pr(g)}^2\,\mathrm{Pr(b)}. \qquad (56)$$

This can be equivalently simplified as

$$\mathrm{Pr(g)}^2\,\mathrm{Pr(b)} < \mathrm{Pr(b)}^2, \text{ then } \mathrm{Pr(g)}^2 < \mathrm{Pr(b)}. \qquad (57)$$

This is verified on condition that $\mathrm{Pr(g)} < \mathrm{Pr(b)}$. According to our model:

$$\mathrm{Pr(g)} = \frac{1}{2}(\mathrm{Q} + 1 - \mathrm{P}) \text{ and } \mathrm{Pr(b)} = \frac{1}{2}(\mathrm{P} + 1 - \mathrm{Q}). \qquad (58)$$

Assumption 2 defines that $\mathrm{P} > \mathrm{Q}$ and $1 - \mathrm{Q} > 1 - \mathrm{P}$; so $\mathrm{Pr(g)} < \mathrm{Pr(b)}$, and this relationship is proved.

2) $\mathrm{Pr}_{\text{naked 1}}^{\text{Wrong}} > \mathrm{Pr}_{\text{Covered}}^{\text{Wrong}}$ is equivalent to:

$$\mathrm{Pr(b)}^2 + 2\,\mathrm{Pr(b)}^2\,\mathrm{Pr(g)} > \mathrm{Pr(g)}\,\mathrm{Pr(b)}^2. \qquad (59)$$

This is also equivalent to

$$\mathrm{Pr(b)}^2 + \mathrm{Pr(b)}^2\,\mathrm{Pr(g)} > 0. \qquad (60)$$

This is true, because $\mathrm{Pr(b)}$ and $\mathrm{Pr(g)}$ are both positive.

3) $\mathrm{Pr}_{\text{naked 1}}^{\text{Non}} < \mathrm{Pr}_{\text{Covered}}^{\text{Non}}$ is equivalent to:

$$0 < \mathrm{Pr(b)}\,\mathrm{Pr(g)}. \qquad (61)$$

This is also evident for $\mathrm{Pr(b)}$ and $\mathrm{Pr(g)}$ are both positive.

Proposition 5

First of all, it is necessary to calculate the bid-ask prices for the fourth trader at whose place the naked trader finishes the pair of two opposite positions. According to lemma 1 and 2, we have:

$$\begin{aligned}
\mathrm{A}_4^{\mathrm{BSB}} &= \mathrm{C}(\sigma_\mathrm{H}) * \mathrm{Pr}(\sigma_\mathrm{H}|\mathrm{B,S,B,b}) + \mathrm{C}(\sigma_\mathrm{L}) * \mathrm{Pr}(\sigma_\mathrm{L}|\mathrm{B,S,B,b}) \\
&= \mathrm{C}(\sigma_\mathrm{H}) * \mathrm{Pr}(\sigma_\mathrm{H}|\mathrm{B,B,B,g}) + \mathrm{C}(\sigma_\mathrm{L}) * \mathrm{Pr}(\sigma_\mathrm{L}|\mathrm{B,B,B,g}) = \mathrm{B}_4^{\mathrm{BBB}} \\
&= \mathrm{C}(\sigma_\mathrm{H}) * \frac{\mathrm{P}^3(1-\mathrm{P})}{\mathrm{P}^3(1-\mathrm{P}) + (1-\mathrm{Q})^3\mathrm{Q}} + \mathrm{C}(\sigma_\mathrm{L}) * \frac{(1-\mathrm{Q})^3\mathrm{Q}}{\mathrm{P}^3(1-\mathrm{P}) + (1-\mathrm{Q})^3\mathrm{Q}}
\end{aligned}$$

$$= C(\overline{\sigma}) + \frac{1}{2}\nabla(\sigma_H - \sigma_L)\frac{P^3(1-P) - (1-Q)^3Q}{P^3(1-P) + (1-Q)^3Q}; \tag{62}$$

$$\begin{aligned} A_4^{BSS} &= C(\sigma_H) * \Pr(\sigma_H|B,S,S,b) + C(\sigma_L) * \Pr(\sigma_L|B,S,S,b) \\ &= C(\sigma_H) * \frac{P^2(1-P)^2}{P^2(1-P)^2 + (1-Q)^2Q^2} + C(\sigma_L) * \frac{(1-Q)^2Q^2}{P^3(1-P) + (1-Q)^2Q^2} \\ &= C(\overline{\sigma}) - \frac{1}{2}\nabla(\sigma_H - \sigma_L)\frac{(1-Q)^2Q^2 - P^2(1-P)^2}{P^2(1-P)^2 + (1-Q)^2Q^2}. \end{aligned} \tag{63}$$

The liar equilibrium is verified if and only if equation (15) is satisfied:

$$B_4^{BBB} - A_2^S > \Pr(b) * \left(B_2^S - A_4^{BSB}\right) + \Pr(g) * \left(B_2^S - A_4^{BSS}\right)$$

By replacing the value of each price in the above equation and regrouping complementary probabilities, we have:

$$\begin{aligned} &[1 + \Pr(b)]\frac{P^3(1-P) - (1-Q)^3Q}{P^3(1-P) + (1-Q)^3Q} + \frac{Q(1-Q) - P(1-P)}{Q(1-Q) + P(1-P)} \\ &\qquad > \Pr(g)\frac{(1-Q)^2Q^2 - P^2(1-P)^2}{P^2(1-P)^2 + (1-Q)^2Q^2} + \frac{P^2 - (1-Q)^2}{P^2 + (1-Q)^2} \end{aligned} \tag{64}$$

It is feasible to divide this inequation into a group of two inequalities.

$$\begin{cases} \Pr(b) * \dfrac{P^3(1-P) - (1-Q)^3Q}{P^3(1-P) + (1-Q)^3Q} > \Pr(g) * \dfrac{(1-Q)^2Q^2 - P^2(1-P)^2}{P^2(1-P)^2 + (1-Q)^2Q^2} \\ \dfrac{P^3(1-P) - (1-Q)^3Q}{P^3(1-P) + (1-Q)^3Q} + \dfrac{Q(1-Q) - P(1-P)}{Q(1-Q) + P(1-P)} > \dfrac{P^2 - (1-Q)^2}{P^2 + (1-Q)^2} \end{cases} \tag{65}$$

The first inequality of (65) can be simplified as follows for $\Pr(b) > \Pr(g)$, which is already proved in proposition 4.

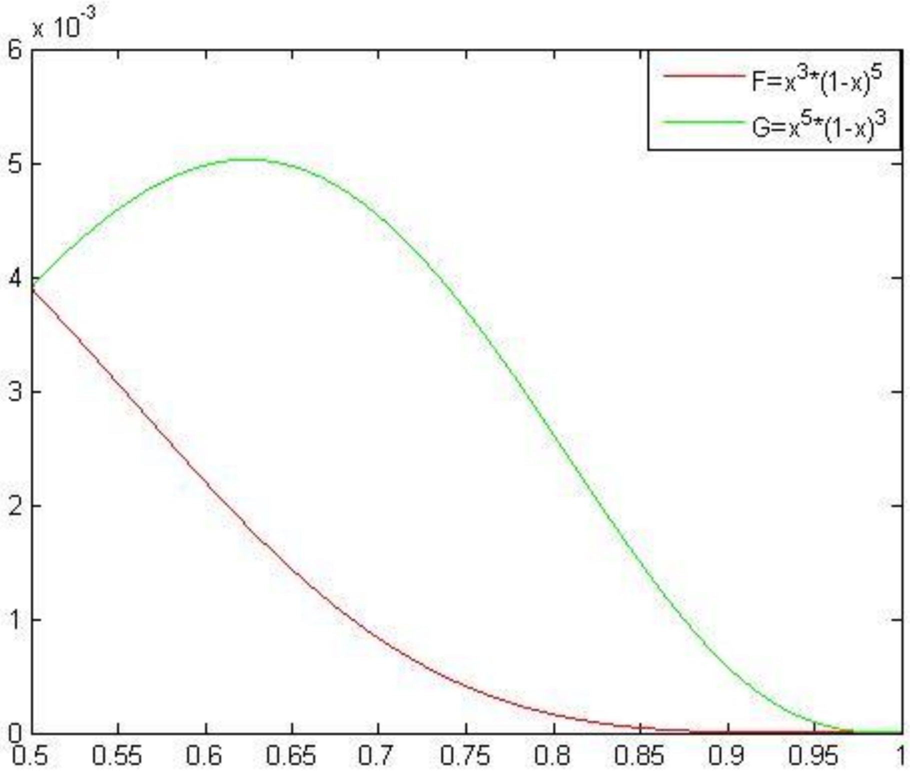
$$\frac{P^3(1-P) - (1-Q)^3Q}{P^3(1-P) + (1-Q)^3Q} > \frac{(1-Q)^2Q^2 - P^2(1-P)^2}{P^2(1-P)^2 + (1-Q)^2Q^2} \tag{66}$$

By eliminating the denominator and regrouping terms, we have:

$$P^5(1-P)^3 > (1-Q)^5Q^3 \tag{67}$$

We define two functions $G = x^5(1-x)^3$ and $F = (1-x)^5x^3$ with $x \in (0.5,1)$. Under the assumption 2 that $P - Q$ is small, it obvious from the beneath figure that $\{G(P) > F(Q)|P - Q \text{ is small}\}$. That is $P^5(1-P)^3 > (1-Q)^5Q^3$.

Figure 5 Function comparison



The second inequality of (65) can be rewritten as follows:

$$\frac{2PQ(1-Q)-2PQ(1-Q)^3(1-P)}{P^4(1-P)^2+PQ(1-P)(1-Q)^3+P^3Q(1-P)(1-Q)+Q^2(1-Q)^4} > \frac{P^2-(1-Q)^2}{P^2+(1-Q)^2} \quad (68)$$

By multiplying the least common multiple of the denominator, we have

$$\begin{aligned} &2P^5(1-P)Q(1-Q)-2PQ(1-Q)^5(1-P) \\ &\quad > P^6(1-P)^2+P^5(1-P)Q(1-Q)+P^2Q^2(1-Q)^4 \\ &\quad - P^4(1-P)^2(1-Q)^2-PQ(1-Q)^5(1-P)-Q^6(1-Q)^2 \qquad (69) \end{aligned}$$

By simplifying and regrouping terms, we have equivalently

$$\begin{aligned} &P^5(1-P)[Q(1-Q)-P(1-P)]-P^2(1-Q)^2[Q^2(1-Q)^2-P^2(1-P)^2] \\ &\quad + Q(1-Q)^5[Q(1-Q)-P(1-P)] > 0 \qquad (70) \end{aligned}$$

By dividing the positive common multiplier $[Q(1-Q)-P(1-P)]$ we have

$$P^5(1-P)+Q(1-Q)^5-P^2(1-Q)^3Q-P^3(1-Q)^2(1-P) > 0 \qquad (71)$$

The terms can be regrouped as follows

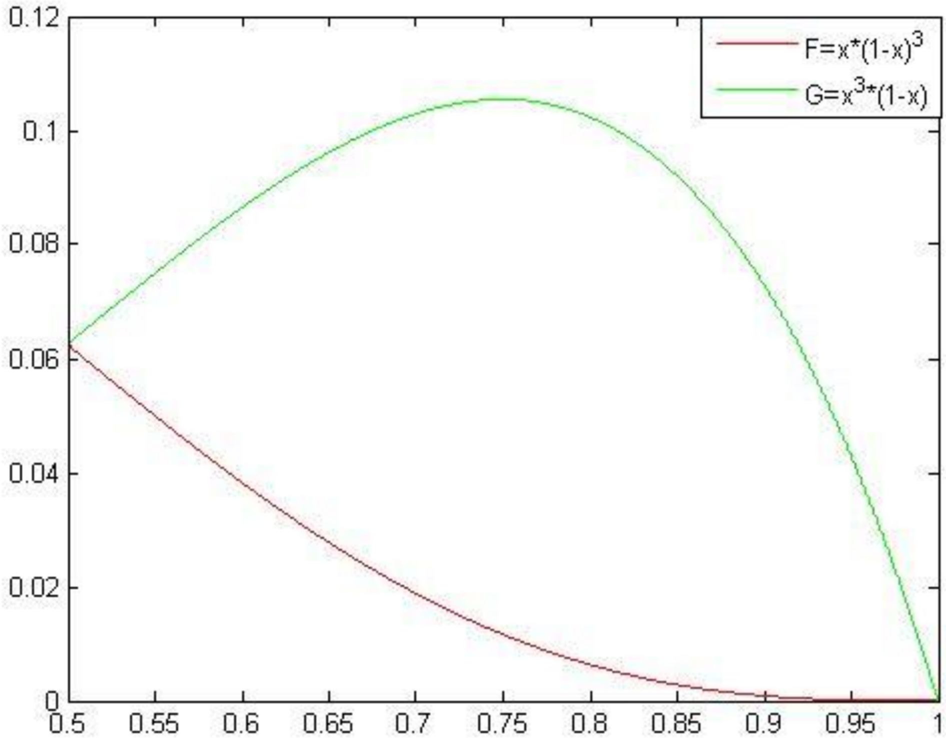
$$[P^3(1-P) - Q(1-Q)^3][P^2 - (1-Q)^2] > 0 \quad (72)$$

According to the basic setting of our model that $P^2 - (1-Q)^2 > 0$, we must have

$$P^3(1-P) - Q(1-Q)^3 > 0 \quad (73)$$

We define again two functions $G = x^3(1-x)$ and $F = (1-x)^3 x$ with $x \in (0.5,1)$.

Figure 6 Function comparison



Under the assumption 2 that $P - Q$ is small, it obvious from the above figure that $\{G(P) > F(Q) | P - Q \text{ is small}\}$. So equation (63) is verified.

Proposition 6

It can be deduced from figure 3 the probabilities of having correct, wrong and no cascade at the equilibrium of reacting naked CDS trading:

$$\begin{cases} \Pr^{\text{Correct}}_{\text{naked 2}} = \Pr(b)^2 + \Pr(g)^2 + \Pr(g)^2 \Pr(b)\,; \\ \Pr^{\text{Wrong}}_{\text{naked 2}} = \Pr(g)\Pr(b)^2 + \Pr(b)\Pr(g)^2\,; \\ \Pr^{\text{No}}_{\text{naked 2}} = \Pr(g)\Pr(b)^2 \end{cases} \quad (74)$$

It is evident that $\mathrm{Pr}_{\text{naked 2}}^{\text{Correct}} = \mathrm{Pr}_{\text{Covered}}^{\text{Correct}}$.

But to verify $\mathrm{Pr}_{\text{naked 2}}^{\text{Wrong}} > \mathrm{Pr}_{\text{Correct}}^{\text{Wrong}}$, it is equivalent to say:

$$\mathrm{Pr}(\mathrm{g})\,\mathrm{Pr}(\mathrm{b})^2 + \mathrm{Pr}(\mathrm{b})\,\mathrm{Pr}(\mathrm{g})^2 > \mathrm{Pr}(\mathrm{b})\,\mathrm{Pr}(\mathrm{g})^2 \tag{75}$$

This can be simplified as:

$$\mathrm{Pr}(\mathrm{g})\,\mathrm{Pr}(\mathrm{b})^2 > 0 \tag{76}$$

This is obvious as the probabilities are both positive.